EIDMA

Lecture 9

- Permutations, permutations with repetitions
- Subsets (combinations)

Recall the last theorem from the previous lecture:

Theorem (The ‘number of functions’ formula)
The number of all functions from an n-element into a k-element set is k^n. In other words, $|Y^X| = |Y|^{|X|}$, where $|X| = n$ and $|Y| = k$.

To simplify notation we will sometimes use $[n] = \{1,2, \ldots ,n\}$ instead of a general n-element set. The last result can be written as

$$\left|[k]^{[n]}\right| = k^{\mathrm{n}}.$$

Notice that you can think about functions from $[n]$ (or any n-element set X) into a k-element set A as n-long sequences of elements of A with no restriction on the number of times a particular element appears in the sequence. Functions are sometimes called *variations with repetitions*.

Also, this is what happens when you pick n elements, one at a time, from a k-element container and you return the chosen element back to the container before you make the next choice (but you keep the record of your choices).

In each case the number of possible results is k^n.

Permutations

Example.

In how many ways can one arrange an exam of 4 tasks chosen from 10 tasks? The order of tasks counts.

Applying the product rule, we can choose one of 10 questions as number 1, then any of the remaining 9 as number two, then 8 choices for number 3 and 7 for number 4. Which means $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

Definition.

An r-*permutation of* n *elements* is a sequence of length r consisting of pairwise different elements of an n-element set.

Equivalently, we can say that an r permutation of n elements is a one-to-one function (an injection) mapping $[r]$ into $[n]$, or that it is an ordered choice of r objects out of n.

Sometimes r-permutations are called *variations without repetitions*.

Definition. (factorial)

For $n > 0$, $n! = \prod_{i=1}^{n} i$; for $n = 0$, $n! = 1$

(inductive definition.: $0! = 1$, $(n+1)! = n!\,(n+1)$).

Theorem. (The number of *r* permutations of *n* elements)

The number of r*-permutations of* n *elements is*

$$n(n-1)(n-2)\ \dots\ (n-r+1)$$

Proof. This an easy consequence of the product rule.

Corollary. The number of n permutations of $[n]$ is $n!$

If $r = n$ (as in the corollary) we simply say "a permutation of n elements" instead of "an n permutation of n elements".

In other words, a permutation is a bijection of $[n]$ into $[n]$, or a total order of an n-element set .

Notice that you can think about r-permutations of elements of an n-element set A as one-to-one functions from $\{1,2, \dots , r\}$ into A. Permutations in this approach are bijections from $\{1,2, \dots , n\}$ into A. The formulae we have developed can be called, respectively,

the number of 1-1 functions from an r-element set into an n-element set

and

the number of bijections from an n-element set into (onto, really) an n-element set.

What if we allow each element from a set to appear in the sequence not just once but a pre-set number of times?
Such arrangements are referred to as
permutations with repetitions
or
permutations of non-distinct objects.

Consider the following example.

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Consider the following example.

Example.

How many different paths can we design from the point (0,0,0) to (3,4,5) if "to take a single step" means "to increment exactly one of the three coordinates by 1"?

Obviously, we need to take 3+4+5 = 12 steps. At each step we must decide which coordinate to increment. If we denote by x, y, z the decision to increment the x, y, z coordinate, respectively, then a single path can be denoted as a 12-long sequence consisting of 3 x's, 4 y's and 5 z's. An example path might look like this (x,y,z,z,x,y,x,y,z,z,y,z).

Example.

Let us print 3 numbered cards, each with an *x* on it, 4 cards with a *y* and 5 with a *z*. Now, the letters are not distinguishable but the cards on which they are printed are – that's what we numbered them for.

1	**2**	**3**	**1**	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**5**
x	x	x	y	y	y	y	z	z	z	z	z

Consider our example path (*x*,*y*,*z*,*z*,*x*,*y*,*x*,*y*,*z*,*z*,*y*,*z*). With our marked cards this sequence may be represented in 3!4!5! ways – you can arrange the *x*-cards in 3! ways, *y*-cards in 4! ways and *z*-cards in 5! ways without affecting the underlying unmarked sequence. The same applies to every sequence of 3 $x's$, 4 $y's$ and 5 $z's$.

3	**2**	**2**	**1**	**2**	**4**	**1**	**1**	**3**	**5**	**3**	**4**
x	y	z	z	x	y	x	y	z	z	y	z

Example. (cont'd)

The total number of permutations of 12 objects is 12!. Those permutations that *differ only in numbers on cards* are considered *equivalent* since they represent the same path from (0,0,0) to (3,4,5). The size of each equivalence class is 3!4!5!. We are interested in the number of paths, which equals the number of the equivalence classes which, in turn, is $\frac{12!}{3!4!5!} = \frac{(3+4+5)!}{3!4!5!}$.

Comprehension. How many different numbers of seven digits can we form out of 2 digits 1, 3 digits 2 and 2 digits 4?

Theorem (Number of permutations with repetitions)
The number of sequences consisting of n_i copies of a_i, where $i = 1,2, \ldots , k$, is $\frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\ldots n_k!}$

Proof. An easy generalization of the last two examples.

SUBSETS

Definition.

The set of all subsets of X is denoted by 2^X or by $\mathcal{P}(X)$.
The set of all k-element subsets of X is denoted by $\mathcal{P}_{\mathrm{k}}(X)$.

In older books you can find the term "combinations" used instead of "subsets". This is where the word "combinatorics" comes from.

Remark.

We can compare sizes (number of elements) of finite sets without actually counting their elements. Two sets, A and B, have the same size (the same number of elements) iff there exists a bijection $f: A \to B$. As a consequence, when we are asked to calculate the size of a set, we can calculate the size of another set and display a bijection mapping one into the other.

Theorem (The number of all subsets of an *n*-element set)
$\left|2^{[n]}\right| = 2^n$, (*equivalently* $|\mathcal{P}([n])| = 2^n$, or $|2^X| = 2^{|X|}$).
Proof 1.
Define a function *f* as follows: for every subset $A \subseteq X$,
$f(A) = \mathbb{1}_A$, where $\mathbb{1}_A(x) = \begin{cases} 1 \text{ for } x \in A \\ 0 \text{ for } x \notin A \end{cases}$, i.e., $\mathbb{1}_A$ is the indicator function of A. $f: \mathcal{P}(X) \to \{0,1\}^X$ and f is a bijection. Hence, the set of all subsets of X has the same size as the set of all functions from X into the 2-element set $\{0,1\}$, i.e., $2^{|X|}$. QED

Comprehension.
Prove that *f* is a bijection.

Proof 2. (Induction on n).
The formula is obviously correct for $n = 0$ and for $n = 1$. Assume it holds for some $n \geq 1$ and consider [n+1]. All subsets of [n+1] come in two types – those who do contain n+1, and those who don't. The number of sets of the first type is obviously the same as the number of all subsets of [n], that is 2^n. How do we get all sets of the second type? We simply add n+1 to each set of the first type, so the number of sets of the second type is also 2^n. Since the two families of sets are disjoint, we can apply the rule of addition and we get $\left|2^{[n+1]}\right| = 2^n + 2^n = 2 \times 2^n = 2^{n+1} = 2^{|[n+1]|} = 2^{n+1}$.
QED

Example.

In how many ways can a group of 10 students designate a nonempty committee that will negotiate their final grades with the professor of Discrete Mathematics? Since we put no restrictions on the size of the committee (other than being nonempty) we calculate the number of all subsets and subtract 1 (for the empty set), i.e. the answer is 1023.

Theorem. (The number of k-element subsets of an n-element set) If $|X| = n$ then, for every k, $0 \leq k \leq n$, we have

$$|\mathcal{P}_k(X)| = \frac{n!}{k!\,(n-k)!}$$

Terminology. *The number* $\frac{n!}{k!(n-k)!}$ *is often denoted by* $\binom{n}{k}$ *and read "n choose k" because it represents the number of ways you can choose k elements out of n.*

Proof.
Let X be an n-element set. Denote the set of all $n!$ permutations of X by S_n. Two permutations are considered k-equivalent iff they have the same elements of X on the first k positions.

Comprehension. *Prove that this is an equivalence relation on* S_n.

The size of every equivalence class is $k!(n-k)!$
Comprehension. *Why?*

This means that the number of equivalence classes is $\frac{n!}{k!(n-k)!}$.

Since the function which assigns to each k-subset A of X the set of all permutations of X with elements of A in the first k places is clearly (?) a bijection, we are done. QED